

Characterization of the finite variation property for a class of stationary increment infinitely divisible processes

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Abstract

We characterize the finite variation property for stationary increment mixed moving averages driven by infinitely divisible random measures. Such processes include fractional and moving average processes driven by Lévy processes, and also their mixtures. We establish two types of zero-one laws for the finite variation property. We also consider some examples to illustrate our results.

Keywords: finite variation; infinitely divisible processes; stationary processes; fractional processes; zero-one laws

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1 Introduction

Processes with stationary, but not necessarily independent, increments have always been of interest in probability and its applications. They are used to model long memory phenomena. Examples include fractional and moving average processes driven by bilateral Lévy processes, as well as their superpositions called mixed fractional and mixed moving average processes, respectively. It has been of interest to determine when such processes are semimartingales and, in particular, when they have locally finite variation. Such questions for Gaussian moving averages were resolved by Knight [13, Theorem 6.5]. Recently, Basse and Pedersen [3] characterized the semimartingale and finite variation properties for stochastic convolutions of non-Gaussian Lévy processes but their arguments do not apply to moving averages. Bender et al. [4] gave necessary and sufficient conditions for square integrable fractional Lévy processes to have sample paths of finite variation and show that the total variation property, for these processes, satisfies a zero-one law.

In this paper we characterize the finite variation property for a wide class of stationary increment infinitely divisible processes that includes fractional Lévy processes, moving averages and mixtures of these processes. We also establish two types of zero-one laws for such processes. Therefore, we extend results of [13] and [4] to a much larger classes of processes but our methods are different. Our work utilizes Banach space techniques, the crucial observation that $BV[0, 1]$, the space of functions of finite variation, is a Banach space of cotype 2, and arguments in the spirit of Hardy and Littlewood [10, Theorem 24].

The paper is organized as follows. In Section 2 we define the class of processes we consider. They are Stationary Increment Mixed Moving Average type (SIMMA for short) processes, see (2.1) and (2.4). In Section 3 we state the main results of this paper. Theorem 3.1 gives sufficient conditions for a SIMMA process to have finite variation. Theorem 3.3, which is the most difficult result of this work, gives necessary conditions. Theorems 3.7–3.8 state the zero-one laws. In Section 4 we determine the finite variation property on examples of processes driven by mixtures of stable random measures and tempered stable random measures. Sections 5 and 6 contain proofs of the main results.

2 Preliminaries

Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a probability space and (V, \mathcal{V}, m) denotes a σ -finite measure space. Let λ be the Lebesgue measure on \mathbb{R} , $\mathcal{B}_0 = \{B \in \mathcal{B}(\mathbb{R}) : \lambda(A) < \infty\}$, and let $\mathcal{V}_0 = \{B \in \mathcal{V} : m(B) < \infty\}$. Consider a stationary increment mixed moving average (SIMMA, for short) process $X = (X_t)_{t \in \mathbb{R}}$ given by

$$X_t = \int_{\mathbb{R} \times V} (f(t-s, v) - f_0(-s, v)) W(ds, dv), \quad t \in \mathbb{R}, \quad (2.1)$$

where $f, f_0 : \mathbb{R} \times V \mapsto \mathbb{R}$ are measurable deterministic functions and W is an independently scattered random measure defined on the σ -ring generated by $\mathcal{B}_0 \times \mathcal{V}_0$ such that for all $A \in \mathcal{B}_0$, $B \in \mathcal{V}_0$ and $u \in \mathbb{R}$,

$$\begin{aligned} & \mathbb{E} e^{iuW(A \times B)} \\ &= \exp \left[\lambda(A) \int_B \left(iu\theta(v) - \frac{1}{2}u^2\sigma^2(v) + \int_{\mathbb{R}} (e^{iux} - 1 - iu\tau(x)) \rho_v(dx) \right) m(dv) \right]. \end{aligned} \quad (2.2)$$

Here $\rho = \{\rho_v : v \in V\}$ is a measurable parametrization of Lévy measures on \mathbb{R} , θ and σ^2 are two measurable functions from V into \mathbb{R} such that $\sigma^2 \geq 0$, and τ is a truncation function on \mathbb{R} , i.e., a bounded function from \mathbb{R} into \mathbb{R} such that $\tau(x) = x + o(x^2)$ as $x \rightarrow 0$. The integral in (2.1) is defined as in Rajput and Rosiński [15]. We further assume that W is purely stochastic, that is

$$m(v : \rho_v(\mathbb{R}) = 0, \sigma^2(v) = 0) = 0. \quad (2.3)$$

Since W is invariant in distribution under the shift on \mathbb{R} , $(X_t)_{t \in \mathbb{R}}$ has stationary increments and thus is continuous in probability, cf. [19]. If V is a one-point space, then the v -component can be removed from (2.1)–(2.2) and W becomes a random measure generated by increments of a two-sided Lévy process that we also denote by W . In this case $(X_t)_{t \in \mathbb{R}}$ is called a stationary increment moving average (SIMA) process written as

$$X_t = \int_{\mathbb{R}} (f(t-s) - f_0(-s)) dW_s, \quad t \in \mathbb{R}, \quad (2.4)$$

If also $f(s) = f_0(s) = s_+^\alpha$ for some $\alpha \in \mathbb{R}$, then $(X_t)_{t \in \mathbb{R}}$ is a fractional Lévy process. If $f_0 \equiv 0$, then $(X_t)_{t \in \mathbb{R}}$ is a moving average. When $f_0 \equiv 0$ in (2.1), then $(X_t)_{t \in \mathbb{R}}$ is a mixed moving average process (cf. [20]). Overall, SIMMA processes cover a large class of stationary increment infinitely divisible processes of interest.

Let $I \subseteq \mathbb{R}$ be an interval, finite or infinite. A function $h: I \rightarrow \mathbb{R}$ is said to be of finite variation, if for all $a, b \in I$ with $a < b$,

$$\|h\|_{BV[a,b]} := \sup_{\substack{a=t_0 < \dots < t_n = b \\ n \in \mathbb{N}}} \sum_{k=1}^n |h(t_k) - h(t_{k-1})| < \infty.$$

For example, if h is absolutely continuous, that is, there exists a locally integrable function \dot{h} such that

$$h(t) - h(u) = \int_u^t \dot{h}(s) ds, \quad u, t \in I, \quad u < t,$$

then h is of finite variation and $\|h\|_{BV[a,b]} = \int_a^b |\dot{h}(s)| ds$.

We will always choose a separable process $X = (X_t)_{t \in \mathbb{R}}$ satisfying (2.1). Since X is continuous in probability, we may and do assume that the set $\mathbb{D} \subset \mathbb{R}$ of dyadic numbers is its separant, see [9]. Then

$$\|X\|_{BV[a,b]} = \sup_{n \in \mathbb{N}} \sum_{i=1}^n |X_{t_i^n} - X_{t_{i-1}^n}| \quad \text{a.s.}, \quad (2.5)$$

where $a = t_0^n < \dots < t_n^n = b$ are such that $\{t_i^n\}_{i=1}^{n-1} \subset \mathbb{D}$ and $\max_{1 \leq i \leq a_n} t_i^n - t_{i-1}^n \rightarrow 0$ as $n \rightarrow \infty$. Similarly, we may view $f_t := f(t - \cdot, \cdot)$ as a stochastic process with respect to some probability measure Q on $\mathbb{R} \times V$ that equivalent to $\lambda \otimes m$. Since X is continuous in probability, from the properties of the stochastic integral (2.1) and (2.3) it follows that $t \mapsto f_t$ is continuous in Q , see [15]. Thus we may and do assume the separability of $(f_t)_{t \in \mathbb{R}}$ with \mathbb{D} being its separant. Consequently, we have

$$\|f\|_{BV[a,b]} = \sup_{n \in \mathbb{N}} \sum_{i=1}^n |f_{t_i^n} - f_{t_{i-1}^n}| \quad \lambda \otimes m\text{-a.e.}, \quad (2.6)$$

where t_i^n are as in (2.5).

3 Main results

3.1 Characterization of finite variation

Here we give closely related sufficient and necessary conditions for SIMMA processes to have paths of finite variation.

Theorem 3.1 (Sufficiency). *Let $X = (X_t)_{t \in \mathbb{R}}$ be a process given by (2.1). Suppose that for m -a.e. v , $f(\cdot, v)$ is absolutely continuous and its derivative $\dot{f}(s, v) = \frac{\partial}{\partial s} f(s, v)$ satisfies the following two conditions*

$$\int_{\mathbb{R}} \int_V (|\dot{f}(s, v)|^2 \sigma^2(v)) m(dv) ds < \infty, \quad (3.1)$$

and

$$\int_{\mathbb{R}} \int_V \int_{\mathbb{R}} (|x \dot{f}(s, v)|^2 \wedge |x \dot{f}(s, v)|) \rho_v(dx) m(dv) ds < \infty. \quad (3.2)$$

Then $(X_t)_{t \in \mathbb{R}}$ has absolutely continuous sample paths a.s. whose total variation is integrable on each finite interval. Moreover, $\lambda \otimes \mathbb{P}$ -a.e.

$$\frac{dX_t}{dt} = \int_{\mathbb{R} \times V} \dot{f}(t - s, v) W(ds, dv), \quad t \in \mathbb{R},$$

where on the right hand side is a well-defined mixed moving average process with paths in L^1 a.s. on each finite interval.

Corollary 3.2. *Let C_f and D_f denote the integrals given by (3.1) and (3.2), respectively. In addition to the assumptions of Theorem 3.1, suppose that W is a mean zero random measure. Then*

$$\mathbb{E} \|X\|_{BV[0,1]} \leq (2/\pi)^{1/2} C_f^{1/2} + (5/4) \max\{D_f, D_f^{1/2}\}.$$

The converse to Theorem 3.1 is more complex due to the vast class of possible random measures W . Assumption (3.3) precludes W having locally finite variation, which necessitates f to have absolutely continuous sections (see Remark 3.6).

Theorem 3.3 (Necessity). *Suppose that X has paths of finite variation a.s. on $[0, 1]$ and that*

$$m\left(v : \int_{-1}^1 |x| \rho_v(dx) < \infty, \sigma^2(v) = 0\right) = 0. \quad (3.3)$$

Then for m -a.e. v , $f(\cdot, v)$ is absolutely continuous, its derivative $\dot{f}(\cdot, v)$ satisfies (3.1) and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (|\dot{f}(s, v)x| \wedge |\dot{f}(s, v)x|^2) (1 \wedge x^{-2}) \rho_v(dx) ds < \infty \quad m\text{-a.e.} \quad (3.4)$$

If, additionally,

$$\limsup_{u \rightarrow \infty} \frac{u \int_{|x| > u} |x| \rho_v(dx)}{\int_{|x| \leq u} x^2 \rho_v(dx)} < \infty \quad m\text{-a.e.} \quad (3.5)$$

then $\dot{f}(\cdot, v)$ satisfies (3.1) and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (|x \dot{f}(s, v)|^2 \wedge |x \dot{f}(s, v)|) \rho_v(dx) ds < \infty \quad m\text{-a.e.} \quad (3.6)$$

Finally, if

$$\sup_{v \in V} \sup_{u > 0} \frac{u \int_{|x| > u} |x| \rho_v(dx)}{\int_{|x| \leq u} x^2 \rho_v(dx)} < \infty \quad (3.7)$$

then $\dot{f}(\cdot, v)$ satisfies (3.1) and (3.2).

Corollary 3.4. *Theorem 3.3 constitutes a complete converse to Theorem 3.1 when (3.3) holds and either (3.5) holds and V is finite or (3.7) holds.*

Surprisingly, it is not easy to find a centered random measure W failing (3.5). Below we will give conditions under which (3.5) or (3.7) hold. Recall that a measure μ on \mathbb{R} is said to be regularly varying if $x \mapsto \mu([-x, x]^c)$ is a regularly varying function; see [5].

Proposition 3.5. *Condition (3.5) is satisfied when one of the following two conditions holds for m -almost every $v \in V$*

(i) $\int_{|x| > 1} x^2 \rho_v(dx) < \infty$ or

(ii) ρ_v is regularly varying at ∞ with index $\beta \in [-2, -1)$.

Condition (3.7) holds when $\rho_v = \rho_0$ for all v and some fixed Lévy measure ρ_0 satisfying (3.5) and such that ρ_0 is regularly varying with index $\beta \in (-2, -1)$ at 0.

Proof of Proposition 3.5. (i). Choose $u_0 = u_0(v) > 0$ such that $\rho_v([-u_0, u_0]) > 0$. Then for all $u > u_0$

$$\frac{u \int_{|x| > u} |x| \rho_v(dx)}{\int_{|x| \leq u} x^2 \rho_v(dx)} \leq \frac{\int_{|x| > u} x^2 \rho_v(dx)}{\int_{|x| \leq u} x^2 \rho_v(dx)} \leq \frac{\int_{|x| > u_0} x^2 \rho_v(dx)}{\int_{|x| \leq u_0} x^2 \rho_v(dx)}.$$

(ii). Using the identities for $u > 0$

$$\begin{aligned} \int_{|x| > u} |x| \rho_v(dx) &= u \rho_v([-u, u]^c) + \int_u^\infty \rho_v([-r, r]^c) dr, \\ \int_{|x| \leq u} x^2 \rho_v(dx) &= 2 \int_0^u r \rho_v([-r, r]^c) dr - u^2 \rho_v([-u, u]^c), \end{aligned}$$

and dividing by $u^2 \rho_v([-u, u]^c)$ in the below fraction gives

$$\frac{u \int_{|x|>u} |x| \rho_v(dx)}{\int_{|x|\leq u} x^2 \rho_v(dx)} = \frac{1 + \int_u^\infty \rho_v([-r, r]^c) dr / (u \rho_v([-u, u]^c))}{2 \int_0^u r \rho_v([-r, r]^c) dr / (u^2 \rho_v([-u, u]^c)) - 1} \rightarrow \frac{1 - (\beta + 1)^{-1}}{2(\beta + 2)^{-1} - 1}$$

as $u \rightarrow \infty$ by Karamata's Theorem [5, Theorem 1.5.11]. (The limit should be understood as 0 when $\beta = -2$.) This shows (3.5).

The proof of the last part of this proposition is similar to the proof of (ii) and thus is omitted. \square

Remark 3.6. As we mentioned earlier, condition (3.3) is in general necessary to deduce that f has absolutely continuous sections. Indeed, let V be a one point space so that W is generated by increments of a Lévy process denoted again by W . If (3.3) is not satisfied, then taking $f = \mathbf{1}_{[0,1]}$ we get that $X_t = W_t - W_{t-1}$ is of finite variation, but f is not continuous.

3.2 Zero-one laws

We distinguish two types of zero-one laws, a global one which always holds and a local one holding only in certain situations.

Theorem 3.7 (Global 0-1). *Let $X = (X_t)_{t \in \mathbb{R}}$ be a process given by (2.1). Then*

$$\mathbb{P}(\|X\|_{BV[a,b]} < \infty \text{ for all } a < b) = 0 \text{ or } 1.$$

Theorem 3.8 (Local 0-1). *Let $a < b$ be fixed reals. Then,*

$$\mathbb{P}(\|X\|_{BV[a,b]} < \infty) = 0 \text{ or } 1 \tag{3.8}$$

provided one of the following conditions is satisfied:

- (a) $f(\cdot, v)$ is of finite variation for m -a.e. v ,
- (b) $\rho_v(\mathbb{R}) = \infty$ for m -a.e. v .

Furthermore, if $\mathbb{P}(\|X\|_{BV[a,b]} < \infty) = 1$, then (a) holds.

Remark 3.9. The following example shows that the local zero-one law does not always hold. Let $[a, b] = [0, 1]$ and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that f has infinite total variation on each subinterval of $[0, 1]$ and $f(x) = 0$ for $x \in [0, 1]^c$. Let

W be a Poisson point process on \mathbb{R} with Lebesgue intensity measure, represented as $W = \sum_j \delta_{\tau_j}$. Consider a moving average process

$$X_t = \int_{\mathbb{R}} f(t-s) W(ds) = \sum_{t-1 \leq \tau_j \leq t} f(t-\tau_j).$$

If the event $W([-1, 1]) = 0$ occurs, then $X_t = 0$ for all $t \in [0, 1]$. Thus

$$\mathbb{P}(\|X\|_{BV[0,1]} < \infty) \geq \mathbb{P}(\|X\|_{BV[0,1]} < \infty, W([-1, 1]) = 0) = e^{-2}.$$

Also, if $W([-1, 0]) = 0$ and $W([0, 1]) = 1$, then $\|X\|_{BV[0,1]} = \infty$. Hence

$$\begin{aligned} \mathbb{P}(\|X\|_{BV[0,1]} < \infty) &= 1 - \mathbb{P}(\|X\|_{BV[0,1]} = \infty) \\ &\leq 1 - \mathbb{P}(\|X\|_{BV[0,1]} = \infty, W([-1, 0]) = 0, W([0, 1]) = 1) = 1 - e^{-2}. \end{aligned}$$

□

4 Examples

In this subsection we will consider two examples of the general set-up. First, in Example 4.1, we will consider the situation where the noise W is of the stable or tempered stable type. More precisely, let ρ_v be given by

$$\rho_v(dx) = (\mathbf{1}_{\{x \geq 0\}} c_1(v) x^{-\alpha(v)-1} + \mathbf{1}_{\{x < 0\}} c_2(v) |x|^{-\alpha(v)-1}) dx, \quad \alpha(v) \in (1 + \epsilon, 2), \quad (4.1)$$

or

$$\rho_v(dx) = (\mathbf{1}_{\{x \geq 0\}} d_1 x^{-\beta-1} e^{-l_1 x} + \mathbf{1}_{\{x < 0\}} d_2 |x|^{-\beta-1} e^{-l_2 |x|}) dx, \quad \beta \in (1, 2), \quad (4.2)$$

where c_1, c_2, α are measurable functions from V into $[0, \infty)$, $d_1, d_2 \geq 0, d_1 + d_2 > 0$, $l_1, l_2 > 0$ and $\epsilon > 0$. Equation (4.1) defines the Lévy measure of a stable distribution with index $\alpha(v)$ and (4.2) is the Lévy measure of a tempered stable distribution with a fixed index β ; see [6].

Example 4.1. Suppose that ρ_v is given by (4.1) or (4.2), and $\sigma^2 = 0$. Then, a SIMMA process $X = (X_t)_{t \in \mathbb{R}}$, given by (2.1), is of finite variation if and only if for m -a.e. v , $f(\cdot, v)$ is absolutely continuous with derivative $\dot{f}(\cdot, v)$ satisfying

$$\begin{cases} \int_V \int_{\mathbb{R}} \left(\frac{c_1(v) + c_2(v)}{2 - \alpha(v)} |\dot{f}(s, v)|^{\alpha(v)} \right) ds m(dv) < \infty & \text{when } \rho \text{ is given by (4.1),} \\ \int_V \int_{\mathbb{R}} \left(|\dot{f}(s, v)|^\beta \wedge |\dot{f}(s, v)|^2 \right) ds m(dv) < \infty & \text{when } \rho \text{ is given by (4.2).} \end{cases}$$

Proof. Let ρ_v is given by (4.1). For $v \in V$, $\int_{-1}^1 |x| \rho_v(dx) = \infty$ and hence (3.3) is satisfied. Using (4.1) a simple calculation shows that

$$\int_{\mathbb{R}} (|xu| \wedge |xu|^2) \rho_v(dx) = C(v) |u|^{\alpha(v)}, \quad u \in \mathbb{R},$$

where

$$C(v) := (c_1(v) + c_2(v)) \left(\frac{1}{\alpha(v) - 1} + \frac{1}{2 - \alpha(v)} \right).$$

A similar calculation shows that

$$u \int_{|x|>u} |x| \rho_v(dx) = K_0(v) \int_{|x|\leq u} x^2 \rho_v(dx), \quad u \geq 0,$$

where $K_0(v) = (2 - \alpha(v))/(\alpha(v) - 1)$, and since $\alpha(v) \in (1 + \epsilon, 2)$ by assumption, (3.7) holds. Hence the result follows by Theorems 3.1+3.3.

Assume that $\rho = \rho_v$ is given by (4.2) and note that $\int_{-1}^1 |x| \rho(dx) = \infty$. In the following we will use the notation $f(u) \sim g(u)$ as $u \rightarrow 0$ (or ∞), if $f(u)/g(u) \rightarrow 1$ as $u \rightarrow 0$ (or ∞). Moreover, we will use the asymptotics of the incomplete gamma functions. We have that

$$\rho([-u, u]^c) \sim (d_1 + d_2) \beta^{-1} u^{-\beta} \quad \text{as } u \rightarrow 0,$$

which by Proposition 3.5 shows that ρ satisfies (3.7), keeping in mind that $\int_{|x|>1} x^2 \rho(dx) < \infty$. From (4.2) we have

$$\int_{\mathbb{R}} (|xu| \wedge |xu|^2) \rho(dx) \sim \begin{cases} C_1 u^\beta & \text{as } u \rightarrow \infty, \\ C_2 u^2 & \text{as } u \rightarrow 0, \end{cases}$$

where $C_1 = (d_1 + d_2)((\beta - 1)^{-1} + (2 - \beta)^{-1})$ and $C_2 = (d_1 l_1^{\beta-2} + d_2 l_2^{\beta-2}) \Gamma(2 - \beta)$, which by Theorems 3.1+3.3 completes the proof. \square

Let $v \mapsto \alpha(v)$ be a measurable function from V into \mathbb{R} and consider $X = (X_t)_{t \in \mathbb{R}}$ of the form

$$X_t = \int_{\mathbb{R} \times V} ((t - s)_+^{\alpha(v)} - (-s)_+^{\alpha(v)}) W(ds, dv), \quad (4.3)$$

where $0^0 := 0$ and $x_+ := \max\{x, 0\}$ for $x \in \mathbb{R}$. We will, as in the rest of this paper, assume that X is well-defined. When V is a one point space, X is called a fractional Lévy process. Thus, a process X of form (4.3) is a superposition of fractional Lévy processes with (possible) different indexes, and will therefore be called a supFLP.

Example 4.2. Let $X = (X_t)_{t \in \mathbb{R}}$ be a supFLP of the form (4.3). If $\sigma^2 = 0$, $\alpha \in [0, \frac{1}{2})$ *m-a.e.* and

$$\int_V \left(\int_{\mathbb{R}} |x|^{\frac{1}{1-\alpha(v)}} \rho_v(dx) \right) \left(\frac{1}{2} - \alpha(v) \right)^{-1} m(dv) < \infty, \quad (4.4)$$

then X is of finite variation. On the other hand, if X is of finite variation, then m -a.e., $\sigma^2 = 0$, $\alpha \in [0, \frac{1}{2})$ and

$$\int_{\mathbb{R}} |x|^{\frac{1}{1-\alpha(v)}} \rho_v(dx) < \infty. \quad (4.5)$$

If, in addition, ρ satisfies (3.7), then (4.4) is satisfied.

To see that the above example follows from Theorems 3.1+3.3 we need the following general facts about supFLPs X of the form (4.3). Process X is of the form (2.1) with $f(s, v) = f_0(s, v) = s_+^{\alpha(v)}$. Since X is well-defined, an application of Rajput and Rosiński [15, Theorem 2.7] shows that

$$\int_V \int_{\mathbb{R}} \int_{\mathbb{R}} \left(|(f(1-s, v) - f_0(-s, v))x|^2 \wedge 1 \right) ds \rho_v(dx) m(dv) < \infty. \quad (4.6)$$

For all $s > 0$ there exists $z = z(s, v) \in [s, s+1]$ such that $f(1+s, v) - f(s, v) = \alpha(v)z^{\alpha(v)-1}$. By (4.6) it follows that $\alpha < \frac{1}{2}$ m -a.e. and since $z^{\alpha-1} \geq (s+1)^{\alpha-1}$, (4.6) shows that

$$\int_V \int_{|x\alpha(v)| > 1} \left(\frac{|\alpha(v)x|^{\frac{1}{1-\alpha(v)}}}{1-2\alpha(v)} \right) \rho_v(dx) m(dv) < \infty,$$

which implies that

$$\int_{|x| > 1} |x|^{\frac{1}{1-\alpha(v)}} \rho_v(dx) < \infty \quad \text{for } m\text{-a.e. } v. \quad (4.7)$$

For $v \in V$, $f(\cdot, v)$ is absolutely continuous if and only if $\alpha(v) > 0$ and in this case $\dot{f}(s, v) = \alpha(v)s_+^{\alpha(v)-1}$. For $\alpha(v) \in (0, \frac{1}{2})$, a simple calculation shows that

$$\int_{\mathbb{R}} (|\dot{f}(s, v)x|^2 \wedge |\dot{f}(s, v)x|) ds = |x|^{\frac{1}{1-\alpha(v)}} \left[|\alpha(v)|^{\frac{1}{1-\alpha(v)}} \left(\frac{1}{\alpha(v)} + \frac{1}{1-2\alpha(v)} \right) \right]. \quad (4.8)$$

The square bracket in (4.8) is, for $\alpha(v) \in (0, \frac{1}{2})$, bounded from above and below by two constants $c_1, c_2 > 0$ times $(\frac{1}{2} - \alpha(v))^{-1}$, which shows that

$$\frac{c_1 |x|^{\frac{1}{1-\alpha(v)}}}{\frac{1}{2} - \alpha(v)} \leq \int_{\mathbb{R}} (|\dot{f}(s, v)x|^2 \wedge |\dot{f}(s, v)x|) ds \leq \frac{c_2 |x|^{\frac{1}{1-\alpha(v)}}}{\frac{1}{2} - \alpha(v)}. \quad (4.9)$$

Proof of Example 4.2. Let $f(s, v) = f_0(s, v) = s_+^{\alpha(v)}$. We may and do consider the following two cases separately: $\alpha(v) = 0$ for all $v \in V$, and $\alpha(v) \neq 0$ for all $v \in V$; use a symmetrization argument in the case where X is of finite variation. If $\alpha(v) = 0$ for all $v \in V$, then, $X_t = W((0, t] \times V)$ is a Lévy process with Lévy measure $\nu(dx) = \rho_v(dx) m(dv)$ and Gaussian component $\int_V \sigma^2(v) m(dv)$. Hence X is of finite variation if and only if $\int_V \int_{\mathbb{R}} |x| \rho_v(dx) m(dv) < \infty$ and $\sigma^2 = 0$ m -a.e., cf. [18, Theorem 21.9]. Thus, in what follows we will assume that $\alpha(v) \neq 0$ for all $v \in V$.

Assume that $\alpha \in (0, \frac{1}{2})$, $\sigma^2 = 0$ m -a.e. and (4.4) is satisfied. For m -a.e. v , $f(\cdot, v)$ is absolutely continuous and by (4.9), $\dot{f}(\cdot, v)$ satisfies (3.2), which by Theorem 3.1 shows that X is of finite variation.

On the other hand, assume that X is of finite variation. By a symmetrization argument we may consider the cases where W is centered Gaussian or has no Gaussian component separately. In the Gaussian case we have $\sigma^2 > 0$ m -a.e. by (2.3), and therefore (3.3) holds. For m -a.e. v , $s \mapsto f(s, v)$ is absolutely continuous with a derivative $\dot{f}(s, v) = \alpha(v)s_+^{\alpha(v)-1}$ satisfying (3.1), cf. Theorem 3.3. Hence $\alpha(v) > 0$ and by (4.9)

$$\int_V \left(\int_0^\infty |s^{\alpha(v)-1}|^2 ds \right) |\alpha(v)|^2 \sigma^2(v) m(dv) < \infty,$$

which implies that $\sigma^2 = 0$ m -a.e. In the purely non-Gaussian case, Rosiński [16, Theorem 4] shows that $f(\cdot, v)$ is of finite variation for m -a.e. v . Hence $\alpha \geq 0$ and by assumption $\alpha > 0$. Thus for m -a.e. v , $f(\cdot, v)$ is absolutely continuous and by (4.9) and the below Remark 5.3 we have

$$\int_{\mathbb{R}} \left(\frac{|x|^{\frac{1}{1-\alpha(v)}}}{1 \vee x^2} \right) \rho_v(dx) < \infty \quad \text{for } m\text{-a.e. } v,$$

which combined with (4.7) show (4.5). Finally, if ρ satisfies (3.7) then Remark 5.3 and (4.9) show that (4.4) is satisfied. This completes the proof. \square

In the special case where V is a one point space, i.e. X is a fractional Lévy process, Example 4.2 shows that X is of finite variation if and only if $\sigma^2 = 0$, $\alpha \in [0, \frac{1}{2})$ and (4.5) is satisfied. This generalizes [3, Corollary 5.4] and parts of [4, Theorem 2.1]. The first mentioned result only considers the necessary part and the second one only considers the centered and square-integrable case.

5 Proofs of Theorems 3.1 and 3.3

We will start by showing Theorem 3.1.

Proof of Theorem 3.1. Let $B = \{v : \dot{f}(\cdot, v) = 0 \text{ } \lambda\text{-a.e.}\}$. By (3.2), $\int_{|x|>1} |x| \rho(dx) < \infty$ m -a.e. on B^c , and since $f(\cdot, v)$ is constant for $v \in B$ we may and do assume that $\int_{|x|>1} |x| \rho(dx) < \infty$ m -a.e. This allows us to write W as $W = W_0 + \mu$, where W_0 is a centered random measure and μ is a deterministic measure. To show that $(X_t)_{t \in \mathbb{R}}$ has absolutely continuous sample paths, define a measurable process $(Y_t^0)_{t \in \mathbb{R}}$ by

$$Y_t^0 = \int_{\mathbb{R} \times V} \dot{f}(t-s, v) W_0(ds, dv). \quad (5.1)$$

By a stochastic Fubini theorem, see [2, Remark 3.2], we have for all $a < b$,

$$\begin{aligned} \int_a^b Y_t^0 dt &= \int_{\mathbb{R} \times V} \left(\int_a^b \dot{f}(t-s, v) dt \right) W_0(ds, dv) \\ &= \int_{\mathbb{R} \times V} (f(b-s, v) - f(a-s, v)) W_0(ds, dv). \end{aligned}$$

Hence by linearity,

$$h(t) := \int_{\mathbb{R} \times V} (f(t-s, v) - f(-s, v)) \mu(ds, dv), \quad t \in \mathbb{R},$$

is well-defined. Using that $h(t) = h(t+u) - h(u)$ for all $u, t \in \mathbb{R}$ and that h is measurable, a standard argument shows that $h(t) = th(1)$. Thus, with $Y_t := h(1) + Y_t^0$, we have with probability 1,

$$X_t = X_0 + \int_0^t Y_u du, \quad t \in \mathbb{R},$$

which proves Theorem 3.1. \square

Proof of Corollary 3.2. Corollary 3.2 follows by the estimates given in Marcus and Rosiński [14], Corollary 1, used on Y_t^0 in (5.1). \square

To prove Theorem 3.3 we need the following Lemmas 5.1–5.2 about general infinitely divisible processes. Let T denote a countable set and $X = (X_t)_{t \in T}$ be an infinitely divisible process without Gaussian component. Let $\mathbb{R}^{(T)}$ be the topological dual of \mathbb{R}^T , which is equipped with the product topology, and let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $\mathbb{R}^{(T)} \times \mathbb{R}^T$. For each $y \in \mathbb{R}^{(T)}$ there exist $n \in \mathbb{N}$, $(\alpha_i)_{i=1}^n \subseteq \mathbb{R}$ and $(t_i)_{i=1}^n \subseteq T$ such that $\langle y, x \rangle = \sum_{i=1}^n \alpha_i x_{t_i}$ for all $x \in \mathbb{R}^T$. Let ν be the Lévy measure of X , that is, ν is a Borel measure on \mathbb{R}^T with $\nu(\{0\}) = 0$ and $\int (1 \wedge x(t)^2) \nu(dx) < \infty$ for all $t \in T$ such that for all $y \in \mathbb{R}^{(T)}$

$$\mathbb{E} e^{i\langle y, X \rangle} = \exp \left(i\langle y, b \rangle + \int_{\mathbb{R}^T} (e^{i\langle y, x \rangle} - 1 - i\langle y, \tau(x) \rangle) \nu(dx) \right), \quad (5.2)$$

where $\tau: \mathbb{R}^T \rightarrow \mathbb{R}^{(T)}$ is given by $\tau(x) = (t \mapsto \tau(x_t))$ and $b \in \mathbb{R}^T$.

Let $h: [0, \infty) \rightarrow [0, \infty)$ be a submultiplicative function, i.e., there exists a constant $c > 0$ such that

$$h(x+y) \leq ch(x)h(y), \quad x, y \geq 0.$$

Assume, moreover, that h is increasing, and for all $\epsilon > 0$ there exists $a_\epsilon > 0$ such that $h(x) \leq a_\epsilon e^{\epsilon x}$ for all $x \geq 0$. Let $h(\infty) = \infty$. The key example is $h: x \mapsto (x \vee 1)^p$ for $p > 0$. If q is a lower semicontinuous pseudonorm on \mathbb{R}^T such that $q(X) < \infty$ a.s., Lemma 2.1 in [17] shows that there exists an $r_0 \in (0, \infty)$ such that $\nu(x \in \mathbb{R}^T : q(x) \geq r_0) < \infty$.

Lemma 5.1. *Let T be a countable set, $X = (X_t)_{t \in T}$ be an infinitely divisible process of the form (5.2) and $q: \mathbb{R}^T \rightarrow [0, \infty]$ be a lower-semicontinuous pseudonorm such that $q(X) < \infty$ a.s. For all $r_0 > 0$ such that $\nu(x \in \mathbb{R}^T : q(x) \geq r_0) < \infty$ we have*

$$\int_{q(x) \geq r_0} h(q(x)) \nu(dx) < \infty \quad \text{if and only if} \quad \mathbb{E} h(q(X)) < \infty.$$

Lemma 5.1 can be proved along the lines of Sato [18, Theorem 25.3] if we use Rosiński and Samorodnitsky [17, Lemma 2.2] instead of [18, Lemmas 2.6–2.7]. The proof is hence omitted.

Lemma 5.2. *Let $N \in \mathbb{N}$, $T = \{k2^{-n} : n \in \mathbb{N}, k = 0, \dots, N2^n\}$ and for $f: T \rightarrow \mathbb{R}$ define*

$$\|f\|_{BV[T]} = \sup_{n \in \mathbb{N}} \sum_{i=1}^{N2^n} |f(k2^{-n}) - f((k-1)2^{-n})|.$$

For all infinitely divisible processes $X = (X_t)_{t \in T}$ of the form (5.2) with $\|X\|_{BV[T]} < \infty$ a.s. we have

$$\int_{\mathbb{R}^T} (1 \wedge \|x\|_{BV[T]}^2) \nu(dx) < \infty.$$

It can be shown that $BV[T]$ is a Banach space of cotype 2, however, it is not separable so Araujo and Giné [1, Theorem 2.2] does not apply to this situation. Instead, to prove Lemma 5.2 we may and do assume that X is symmetric. Combining Rajput and Rosiński [15, Theorem 4.11] and Rosiński [16, Proposition 2] show that X has a series representation. By using the series representation, Lemma 5.2 follows along the lines of Proposition 5.6 in Basse and Pedersen [3].

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. Using (2.5) we have by the monotonicity,

$$\mathbb{E}\|X\|_{BV[0,1]} = \sup_{n \in \mathbb{N}} \sum_{k=1}^n \mathbb{E}|X_{k2^{-n}} - X_{(k-1)2^{-n}}| = \sup_{n \in \mathbb{N}} \mathbb{E}|2^n(X_{2^{-n}} - X_0)|. \quad (5.3)$$

For any $v \in V$ let

$$\xi_v(u) := \int_{\mathbb{R}} (|ux|^2 \wedge |ux|) \rho_v(dx).$$

Then ξ_v is symmetric, strictly increasing, and comparable with a convex function $\tilde{\xi}_v$ given by

$$\tilde{\xi}_v(u) = \int_{\mathbb{R}} (|ux|^2 \mathbf{1}_{|ux| \leq 1} + (2|ux| - 1) \mathbf{1}_{|ux| > 1}) \rho_v(dx).$$

Indeed, $\tilde{\xi}_v(u)/2 \leq \xi_v(u) \leq \tilde{\xi}_v(u)$, $u \geq 0$. By Corollary 1.1 in Marcus and Rosiński [14]

$$\frac{1}{4} \min\{I_n, I_n^{1/2}\} \leq \mathbb{E}|2^n(X_{2^{-n}} - X_0)| \leq \frac{5}{4} \max\{I_n, I_n^{1/2}\}, \quad (5.4)$$

where

$$I_n = \int_{\mathbb{R}} \int_V \xi_v(f_n(s, v)) m(dv) ds \quad \text{and} \quad f_n(s, v) = 2^n[f(2^{-n} - s, v) - f(-s, v)]. \quad (5.5)$$

In view of (5.3) and (5.4),

$$\mathbb{E}\|X\|_{BV[0,1]} < \infty \quad \text{if and only if} \quad \sup_{n \in \mathbb{N}} I_n < \infty. \quad (5.6)$$

Since $(X_t)_{t \in \mathbb{R}}$ is of finite variation so is its symmetrization. Therefore, we may and will assume that $\{\rho_v : v \in V\}$ are symmetric Lévy measures on \mathbb{R} . Since the dyadic numbers are a separant for $(f(t - \cdot, \cdot))_{t \in \mathbb{R}}$, we have by Lemma 5.2 for every $t > 0$

$$\int_{\mathbb{R}} \int_V \int_{\mathbb{R}} (1 \wedge \|xf(\cdot - s, v)\|_{BV[0,t]}^2) \rho_v(dx) m(dv) ds < \infty \quad (5.7)$$

which implies that there is a subset $V_0 \in \mathcal{V}$ with $m(V \setminus V_0) = 0$ such that for every $v \in V_0$

$$\int_{\mathbb{R}} (1 \wedge \|f(\cdot - s, v)\|_{BV[0,t]}^2) ds < \infty. \quad (5.8)$$

We will show that

$$k^*(v) := \sup_{s \in \mathbb{R}} \|f(\cdot - s, v)\|_{BV[0,1]} < \infty, \quad v \in V_0. \quad (5.9)$$

To do this notice that

$$\|f(\cdot - s, v)\|_{BV[0,t]} = \|f(\cdot, v)\|_{BV[-s, t-s]} = k(t - s, v) - k(-s, v),$$

where

$$k(u, v) = \begin{cases} \|f(\cdot, v)\|_{BV[0,u]} & \text{if } u \geq 0, \\ -\|f(\cdot, v)\|_{BV[u,0]} & \text{if } u < 0. \end{cases}$$

For each $v \in V_0$, $u \mapsto k(u, v)$ is a nondecreasing function. To show (5.9) fix $v \in V_0$ and let us for the moment suppress v . Let $h(s) = |k(1 - s) - k(-s)|$. For contradiction assume that h is unbounded. Since h is locally bounded there exists a sequence $(a_n)_{n \in \mathbb{N}}$ converging to either ∞ or $-\infty$ (say, ∞) such that $h(a_n) \geq 1$ for all $n \in \mathbb{N}$. By passing to a subsequence we may assume that $a_n + 1 \leq a_{n+1}$ for all $n \in \mathbb{N}$. For $s \in [a_n, a_n + 1]$ we have

$$k(2 - s) - k(-s) \geq k(1 - a_n) - k(-a_n) = h(a_n) \geq 1.$$

Thus,

$$\int_{\mathbb{R}} (1 \wedge [k(2 - s) - k(-s)]^2) ds \geq \sum_{n=1}^{\infty} \int_{a_n}^{a_n+1} (1 \wedge [k(2 - s) - k(-s)]^2) ds \geq \sum_{n=1}^{\infty} 1 = \infty,$$

which contradicts (5.8) and completes the proof of (5.9).

Assume that ρ satisfies (3.5), and let us show that for m -a.e. v , $f(\cdot, v)$ is absolutely continuous with a derivative $\dot{f}(\cdot, v)$ satisfying (3.1) and (3.6). By having shown this, the two other cases in Theorem 3.3 follow as a consequence, as we will see at the end of the proof. By (3.5) there exist two measurable functions $u_0 : V \rightarrow [0, \infty)$ and $K_0 : V \rightarrow (0, \infty)$ such that for m -a.e. v

$$u \int_{|x| > u} |x| \rho_v(dx) \leq K_0(v) \int_{|x| \leq u} x^2 \rho_v(dx) \quad \text{for } u \geq u_0(v),$$

which implies that

$$\int_{|ux|>1} |xu| \rho_v(dx) \leq K_0(v) \int_{\mathbb{R}} (|xu|^2 \wedge 1) \rho_v(dx) \quad \text{for } |u| \leq 1/u_0(v). \quad (5.10)$$

For arbitrary but fixed $k \in \mathbb{N}$ define

$$V_k = \{v \in V_0 : k^*(v) \leq k, K_0(v) \leq k, u_0(v) \leq k\},$$

and let $(X_t^k)_{t \in \mathbb{R}}$ be given by

$$X_t^k = \int_{\mathbb{R} \times V_k} (f(t-s, v) - f_0(-s, v)) W(ds, dv), \quad t \in \mathbb{R}.$$

By a symmetrization argument, $\|X^k\|_{BV[0,1]} < \infty$ a.s. We will show that

$$\mathbb{E}\|X^k\|_{BV[0,1]} < \infty. \quad (5.11)$$

To this end it is enough, according to Lemma 5.1, to prove that

$$\int_{\{(s,v,x) \in \mathbb{R} \times V_k \times \mathbb{R} : \|xf(\cdot-s, v)\|_{BV[0,1]} > k^2\}} \|xf(\cdot-s, v)\|_{BV[0,1]} ds \rho_v(dx) m(dv) < \infty. \quad (5.12)$$

For $v \in V_k$,

$$\|f(\cdot-s, v)\|_{BV[0,1]} k^{-2} \leq k^*(v) k^{-2} \leq k^{-1} \leq 1/u_0(v).$$

Thus applying (5.10) on $u = \|f(\cdot-s, v)\|_{BV[0,1]} k^{-2}$ shows that the left-hand side of (5.12) is less than or equal to

$$k^3 \int_{\mathbb{R}} \int_{V_k} \int_{\mathbb{R}} \left(\|xf(\cdot-s, v)\|_{BV[0,1]}^2 \wedge 1 \right) \rho_v(dx) m(dv) ds$$

which is finite by (5.7). This completes the proof of (5.11).

Since $\mathbb{E}\|X^k\|_{BV[0,1]} < \infty$, (5.6) shows that

$$\sup_{n \in \mathbb{N}} \int_{V_k} \int_{\mathbb{R}} \xi_v(f_n(s, v)) ds m(dv) < \infty, \quad (5.13)$$

where f_n are given as in (5.5). Set

$$J = \{v \in V : \int_{-1}^1 |x| \rho_v(dx) = \infty\},$$

and choose $(A_k)_{k \in \mathbb{N}} \subseteq \mathcal{V}$ such that $A_k \uparrow V$ and $m(A_k) < \infty$ for all $k \in \mathbb{N}$, and let $\lambda_k := \lambda|_{[-k, k]}$ and $m_k := m|_{A_k \cap V_k \cap J}$. Note that $\lambda_k \otimes m_k$ is a finite measure. For $v \in J$, we have by the monotone convergence theorem that

$$\frac{\xi_v(x)}{x} = \int_{\mathbb{R}} (|u^2 x| \wedge |u|) \rho_v(du) \nearrow \int_{\mathbb{R}} |u| \rho_v(du) = \infty \quad \text{as } x \nearrow \infty.$$

Hence for all $k \in \mathbb{N}$ there exists, by Egorov's Theorem (see [12], Chapter 9, Theorem 1), $B_k \in \mathcal{V}$ with $m_k(B_k^c) < 1/k$ such that for all $C > 0$ there exists $K > 0$ such that for all $v \in B_k$, $\inf_{x>K} (\xi_v(x)/x) \geq C$. With $\tilde{m}_k := m_k|_{B_k}$ we have that

$$\begin{aligned} \int_{|f_n|>K} |f_n| d(\lambda_k \otimes \tilde{m}_k) &= \int_{|f_n|>K} \xi_v(f_n(s, v)) \frac{|f_n(s, v)|}{\xi_v(f_n(s, v))} \lambda_k(ds) \tilde{m}_k(dv) \\ &\leq \int_{|f_n|>K} \xi_v(f_n(s, v)) \left(\sup_{x>K} \frac{x}{\xi_v(x)} \right) \lambda_k(ds) \tilde{m}_k(dv) \leq \frac{1}{C} \int \xi_v(f_n(s, v)) \lambda_k(ds) \tilde{m}_k(dv), \end{aligned}$$

which shows that

$$\sup_{n \in \mathbb{N}} \int_{|f_n|>K} |f_n| d(\lambda_k \otimes \tilde{m}_k) \leq \frac{1}{C} \sup_{n \in \mathbb{N}} \int \xi_v(f_n(s, v)) \lambda_k(ds) \tilde{m}_k(dv). \quad (5.14)$$

By (5.13)–(5.14) we conclude that $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable with respect to $\lambda_k \otimes \tilde{m}_k$. Therefore, by the Dunford-Pettis Theorem, see [7], IV.8, Corollary 11, there exists a subsequence $(n_j)_{j \in \mathbb{N}}$ and a $h \in L^1(\lambda_k \otimes \tilde{m}_k)$ such that $\lim_j f_{n_j} = h$ in $\sigma(L^1, L^\infty)$. For all $A \in \mathcal{V}$ with $A \subseteq A_k \cap V_k \cap B_k \cap J$ and for $(\lambda \otimes \lambda)$ -a.e. (s, t) with $-k \leq s < t \leq k$,

$$\begin{aligned} \int_A \left(\int_s^t h(u, v) du \right) m(dv) &= \lim_{j \rightarrow \infty} \int_A \left(\int_s^t f_{n_j}(u, v) du \right) m(dv) \\ &= \lim_{j \rightarrow \infty} 2^{n_j} \left[\int_{s+2^{n_j}}^{t+2^{n_j}} \left(\int_A f(u, v) m(dv) \right) du - \int_s^t \left(\int_A f(u, v) m(dv) \right) du \right] \\ &= \lim_{j \rightarrow \infty} 2^{n_j} \int_t^{t+2^{n_j}} \left(\int_A f(u, v) m(dv) \right) du - \lim_{j \rightarrow \infty} 2^{n_j} \int_s^{s+2^{n_j}} \left(\int_A f(u, v) m(dv) \right) du \\ &= \int_A (f(t, v) - f(s, v)) m(dv). \end{aligned} \quad (5.16)$$

Since (5.15)–(5.16) is satisfied for k arbitrary we have for $(\lambda \otimes \lambda \otimes m)$ -a.e. $(t, s, v) \in \mathbb{R} \times \mathbb{R} \times J$,

$$f(t, v) - f(s, v) = \int_s^t h(u, v) du,$$

which shows that for m -a.e. $v \in J$, $f(\cdot, v)$ is absolutely continuous with derivative $h(\cdot, v)$.

Let $G = \{v \in V : \sigma^2(v) > 0\}$. By Gaussianity, Fernique [8] shows that $\mathbb{E}\|X\|_{BV[0,1]} < \infty$. Let f_n be given by (5.5). As in (5.3) we have that

$$\mathbb{E}\|X\|_{BV[0,1]} = \sup_{n \in \mathbb{N}} (2^n \mathbb{E}|X_{1/2^n} - X_0|) = \sqrt{\frac{2}{\pi}} \sup_{n \in \mathbb{N}} (2^n \|X_{1/2^n} - X_0\|_{L^2}) \quad (5.17)$$

$$= \sqrt{\frac{2}{\pi}} \left(\sup_{n \in \mathbb{N}} \int_V \int_{\mathbb{R}} |f_n(s, v)|^2 \sigma^2(v) ds m(dv) \right)^{1/2}, \quad (5.18)$$

where in the second equality we have used the identity $\|U\|_{L^1} = (2/\pi)^{1/2}\|U\|_{L^2}$ for centered Gaussian random variables U . Let $\mu(ds, dv) = ds \sigma^2(v) m(dv)$ be a measure on $\mathbb{R} \times V$. Since $L^2(\mu)$ is a Hilbert space and $\{f_n : n \in \mathbb{N}\}$ is bounded in $L^2(\mu)$ by (5.17)–(5.18), there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ and a $g \in L^2(\mu)$ such that $\{f_{n_k}\}$ converges to g in $\sigma(L^2, L^2)$, see [7, IV.4, Corollary 7]. As in (5.15)–(5.16) it follows that for m -a.e. $v \in G$, $f(\cdot, v)$ is absolutely continuous with derivative g .

By assumptions $m(G^c \cap J^c) = 0$, and hence for m -a.e. v , $f(\cdot, v)$ is absolutely continuous; let $\dot{f}(\cdot, v)$ denote its derivative. Since $\dot{f} \in L^2(\mu)$, (3.1) follows and we only need to show (3.6). Since for m -a.e. v , $f(\cdot, v)$ is absolutely continuous with derivative $\dot{f}(\cdot, v)$ we have that $f_n \rightarrow \dot{f} \lambda \otimes m$ -a.e., and by continuity of $s \mapsto \xi_v(s)$, it follows that $\xi_v(f_n(s, v)) \rightarrow \xi_v(\dot{f}(s, v))$ for $\lambda \otimes m$ -a.e. (s, v) . Thus, by Fatou's Lemma and (5.13),

$$\int_{V_k} \int_{\mathbb{R}} \xi_v(\dot{f}(s, v)) ds m(dv) \leq \liminf_{n \rightarrow \infty} \int_{V_k} \int_{\mathbb{R}} \xi_v(f_n(s, v)) ds m(dv) < \infty,$$

which shows (3.6). This completes the proof under the assumption (3.5).

In the general situation, define two (positive) Lévy measures ρ_v^1 and ρ_v^2 by

$$\rho_v^1(dx) = \frac{1}{1 \vee x^2} \rho_v(dx) \quad \text{and} \quad \rho_v^2 = \rho_v - \rho_v^1, \quad v \in V,$$

and let X^1 and X^2 be two independent processes defined as X with ρ replaced by $\rho^1 = \{\rho_v^1 : v \in V\}$ and $\rho^2 = \{\rho_v^2 : v \in V\}$, respectively. Since $X =_d X^1 + X^2$, a symmetrization argument shows that X^1 is of finite variation. Moreover, since $\int_{|x|>1} x^2 \rho_v^1(dx) = \rho_v([-1, 1]^c) < \infty$, Proposition 3.5 shows that ρ^1 satisfies (3.5), and hence (3.1) and (3.4) follow by the above.

Finally, assume that ρ satisfies (3.7). This yields the existence of a real constant $C_0 > 0$ such that for all $u > 0$ and $v \in V$

$$\int_{|xu|>1} |xu| \rho_v(dx) \leq C_0 \int_{\mathbb{R}} (|xu|^2 \wedge 1) \rho_v(dx).$$

Hence for all $r > 0$,

$$\begin{aligned} & \int_{\{(x,v,s) \in \mathbb{R} \times V \times \mathbb{R} : \|xf(\cdot - s, v)\|_{BV[0,1]} > 1\}} \|xf(\cdot - s, v)\|_{BV[0,1]} \rho_v(dx) m(dv) ds \\ & \leq C_0 \int_{\mathbb{R}} \int_V \int_{\mathbb{R}} (1 \wedge \|xf(\cdot - s, v)\|_{BV[0,1]}^2) \rho_v(dx) m(dv) ds < \infty, \end{aligned}$$

which by Lemma 5.1 shows that $\mathbb{E}\|X\|_{BV[0,1]} < \infty$. By arguing as above, (3.2) follows. \square

Remark 5.3. In the proof of Theorem 3.3 we only used the assumption (3.3) to conclude that $f(\cdot, v)$ is absolutely continuous for m -a.e. v . Thus if we know that $f(\cdot, v)$ is absolutely continuous for m -a.e. v then Theorem 3.3 remains valid even without the assumption (3.3).

6 Proofs of Theorems 3.7 and 3.8

Proof of Theorem 3.7. Recall that \mathbb{D} denotes the set of dyadic numbers in \mathbb{R} . Consider $\mathbb{R}^{\mathbb{D}}$ as a locally convex separable linear metric space and consider $X_{\mathbb{D}} := (X_t)_{t \in \mathbb{D}}$ as a random variable in $\mathbb{R}^{\mathbb{D}}$. For each $N \in \mathbb{N}$, define

$$H_N = \left\{ h \in \mathbb{R}^{\mathbb{D}} : \sup_{n \in \mathbb{N}} \sum_{i=1}^{2N2^n} |h(r_{n,i}^N) - h(r_{n,i-1}^N)| < \infty \right\},$$

where $r_{n,i}^N = i2^{-n} - N$, and let $H = \bigcap_{N=1}^{\infty} H_N$. By (2.5)

$$\mathbb{P}(\|X\|_{BV[a,b]} < \infty \text{ for all } -\infty < a < b < \infty) = \mathbb{P}(X_{|\mathbb{D}} \in H).$$

Let ν be the Lévy measure of $X_{\mathbb{D}}$. We have

$$\begin{aligned} \nu(H_N^c) &= \int_{\mathbb{R}} \int_{\mathbb{R} \times V} \mathbf{1}_{H_N^c}(xf(\cdot - s, v)) \rho_v(dx) m(dv) ds \\ &= \int_{\mathbb{R}} \int_V \rho_v(\mathbb{R}) \mathbf{1}_{H_N^c}(f(\cdot - s, v)) m(dv) ds \end{aligned} \quad (6.1)$$

because $\rho_v(\{0\}) = 0$. By (2.6) we also have

$$\|f(\cdot - s, v)\|_{BV[-N,N]} = \sup_{n \in \mathbb{N}} \sum_{i=1}^{2N2^n} \left| f(r_{n,i}^N - s, v) - f(r_{n,i-1}^N - s, v) \right| \quad \lambda \otimes m\text{-a.e.}$$

Consider the set

$$A = \{v : \rho_v(\mathbb{R}) > 0 \text{ and } \|f(\cdot, v)\|_{BV[-M,M]} = \infty \text{ for some } M \in \mathbb{N}\}.$$

If $m(A) = 0$ then $\nu(H_N^c) = 0$ for every N , and so $\nu(H^c) = \lim_{N \rightarrow \infty} \nu(H_N^c) = 0$. From Janssen [11, Theorem 9], we get $\mathbb{P}(X_{|\mathbb{D}} \in H) = 0$ or 1.

Suppose now that $m(A) > 0$, so that $m(A_M) > 0$ for some $M \in \mathbb{N}$, where

$$A_M = \{v : \rho_v(\mathbb{R}) > 0 \text{ and } \|f(\cdot, v)\|_{BV[-M,M]} = \infty\}. \quad (6.2)$$

For every $N > M$ and all $(s, v) \in [M - N, N - M] \times A_M$ we have

$$\|f(\cdot - s, v)\|_{BV[-N,N]} \geq \|f(\cdot, v)\|_{BV[-M,M]} = \infty,$$

which combined with (6.1) gives

$$\nu(H_N^c) \geq 2(N - M) \int_{A_M} \rho_v(\mathbb{R}) m(dv).$$

Thus $\nu(H^c) = \lim_{N \rightarrow \infty} \nu(H_N^c) = \infty$. By Janssen [11, Theorem 10], $\mathbb{P}(X_{|\mathbb{D}} \in H) = 0$. This completes the proof. \square

Proof of Theorem 3.8. Fix $a < b$ and define

$$H = \left\{ h: \mathbb{D} \rightarrow \mathbb{R} : \sup_{n \in \mathbb{N}} \sum_{i=1}^{k_n} |h(r_{n,i}) - h(r_{n,i-1})| < \infty \right\},$$

where $\{r_{n,i}\}$ are a dyadic partitions of $[a, b]$ such that $\max_{1 \leq i \leq a_n} (r_{i,n} - r_{i-1,n}) \rightarrow 0$ as $n \rightarrow \infty$. As in (6.1) we show that

$$\nu(H^c) = \int_{\mathbb{R}} \int_V \rho_v(\mathbb{R}) \mathbf{1}_{H^c}(f(\cdot - s, v)) m(dv) ds. \quad (6.3)$$

If (a) holds then $\nu(H^c) = 0$ and the zero-one law holds by the same argument as in the previous theorem.

Assume (b). Let $m(A_M) > 0$ for some $M > 0$, where A_M is given by (6.2). Then there exists a subinterval $[c, d] \subset [-M, M]$ with $d - c < (b - a)/2$ such that $m(B) > 0$, where

$$B := \{v : \rho_v(\mathbb{R}) > 0 \text{ and } \|f(\cdot, v)\|_{BV[c,d]} = \infty\}.$$

For all $(s, v) \in [a - c, b - d] \times B$ we have

$$\|f(\cdot - s, v)\|_{BV[a,b]} \geq \|f(\cdot, v)\|_{BV[-M,M]} = \infty,$$

which combined with (6.3) gives

$$\nu(H^c) \geq \frac{b - a}{2} \int_B \rho_v(\mathbb{R}) m(dv) = \infty.$$

By the same argument as in the proof of Theorem 3.7 we infer that probability in (3.8) is zero. If $m(A_M) = 0$ for all $M \in \mathbb{N}$, then $\nu(H^c) = 0$. We conclude, as above, that the probability in (3.8) is 0 or 1. \square

Finally, let us note that the methods of proofs of Theorems 3.7–3.8 will work if we replace the total variation norm $\|\cdot\|_{BV[a,b]}$ by some wider class of seminorms of sample paths.

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